High-Order Finite Elements for Inhomogeneous Acoustic Guiding Structures

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Abstract—**Silvester's high-order finite-element formulation for potential problems is extended to enable the analysis of acoustic wave propagation in lossless isotropic, uniform, and inhomogeneous guiding structures. The formulation allows a large class of problems to be solved using elements of any desired order, with only minimal computer coding. Three examples are cited—one involving a simple homogeneous region having an analytic solution, and two inhomogeneous problems. Good agreement with other methods and with limiting cases is obtained in each case.**

I. Introduction

THE PROPAGATION characteristics of surface acoustic waveguides and couplers are more difficult to obtain than their electromagnetic (EM) counterparts. Even a structure as simple as a free rectangular bar is not amenable to exact solution [1].

The various structures which have been proposed for guiding surface acoustic waves have been analyzed using a variety of approximation procedures [2]–[4], each having its own particular limitations. Numerical methods of solution involving field discretization, whose advantages (and limitations) are well known in other contexts [5], [6], are beginning to play an important role in the analysis of these structures [7]–[9]. It is worth noting here that the advantages and limitations of the above methods tend to make them complementary rather than redundant.

High-order finite-element methods have been used extensively in solid mechanics [13], [15]. However, there are two aspects which make a new formulation desirable for acoustic guiding analysis: 1) the traveling-wave nature of the solutions of interest, and 2) the desirability of a high-order formulation which separates the geometric and material properties of a particular problem from the essential features of the variational principle.

The purpose of this paper is to develop such a formulation as an extension of Silvester's systematic approach [6], emphasizing the application to nonhomogeneous structures.

In Section II the appropriate variational principle is briefly stated, followed in Section III by a discussion of the relevant boundary and interface conditions. The finite-element formulation is developed in Section IV, and in Section V some examples illustrate the formulation.

II. Variational Principle

A complete mode set can be found for any structure, consisting of lossless isotropic materials uniform in the z direction, by taking a displacement function of the form

$$\boldsymbol{u} = \boldsymbol{u}(u_x, u_y, ju_z) \exp\{j(\omega t - \beta z)\}$$

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where u_x, u_y, and u_z are real functions of the x, y Cartesian coordinates transverse to the direction of propagation, for real β.

The Lagrange density function $\mathcal{L}^{(m)}$ for the problems being considered may then be written in the convenient form

$$\begin{aligned}\mathcal{L}^{(m)} &= \tfrac{1}{2}\rho^{(m)}\omega^2(u_x^2 + u_y^2 + u_z^2)\\ &\quad - \tfrac{1}{2}\{(\lambda^{(m)} + 2\mu^{(m)})(|S_{xx}|^2 + |S_{yy}|^2 + |S_{zz}|^2)\\ &\quad + 2\lambda^{(m)} \operatorname{Re}(S_{xx}S_{yy}^* + S_{yy}S_{zz}^* + S_{zz}S_{xx}^*)\\ &\quad + 4\mu^{(m)}(|S_{xy}|^2 + |S_{yz}|^2 + |S_{zx}|^2)\} \qquad (1)\end{aligned}$$

where $\lambda^{(m)}$ and $\mu^{(m)}$ are the Lame constants, $\rho^{(m)}$ is the mass density, and the superscripts denote the mth medium, assumed homogeneous and isotropic. The strain components S in (1) are given by

$$\begin{aligned}S_{xx} &= \frac{\partial u_x}{\partial x}, \quad S_{yy} = \frac{\partial u_y}{\partial y}, \quad S_{zz} = \beta u_z\\ S_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = S_{yx}\\ S_{yz} &= \frac{1}{2}j\left(\frac{\partial u_z}{\partial y} - \beta u_y\right) = S_{zy}\\ S_{zx} &= \frac{1}{2}j\left(\frac{\partial u_z}{\partial x} - \beta u_x\right) = S_{xz} \qquad (2)\end{aligned}$$

where the factor $\exp\{j(\omega t - \beta z)\}$ is omitted in each case. Note that (1) is equivalent to the forms given in [10] and [11], except that complex phasor quantities are taken into account by conjugating all strain tensor components in the potential energy term.

The $\exp\{j\omega t\}$ variation for all t means that initial conditions and past history are of no account. Furthermore, vanishing stress is prescribed on the boundary of the structure and body forces are neglected. Then for a unit length in the direction of propagation, the classical form of Hamilton's principle [10] reduces to

$$\delta \sum_m L^{(m)} = \sum_m \delta L^{(m)} = 0$$

where

$$\delta L^{(m)} = \delta \int_{S^{(m)}} \mathcal{L}^{(m)}\, dS. \qquad (3)$$

On taking the variation and integrating by parts, (3) becomes

$$\sum_m \left\{ \int_{S^{(m)}} [(\lambda^{(m)} + \mu^{(m)})\nabla\nabla\cdot\boldsymbol{u} + \mu^{(m)}\nabla^2\boldsymbol{u} - \rho^{(m)}\boldsymbol{u}]\cdot\delta\boldsymbol{u}\, dS - \int_{B'^{(m)}} T_n^{(m)}\cdot\delta\boldsymbol{u}\, ds\right\} = 0 \quad (4)$$

where ∇ is the vector operator $e_x(\partial/\partial x)+e_y(\partial/\partial y)+e_z(-j\beta)$, $S^{(m)}$ is the surface formed by taking a transverse cross section of the mth medium, $B^{(m)}$ is the boundary of $S^{(m)}$, and $T_n^{(m)}$ is the surface traction on $B^{(m)}$. Summation is taken over all media and $\boldsymbol{u}$ and its variation $\delta\boldsymbol{u}$ are constrained to be continuous everywhere.

The surface integrals in (4) yield the equations of motion within a medium and the contour integrals yield the natural boundary conditions.

III. Boundary Conditions

There are five boundary conditions to be considered on a transverse cross section of the structures of interest: 1) a solid–air interface, regarded as a traction-free boundary; 2) a rigid boundary; 3) a line of symmetry; 4) a line of antisymmetry; and 5) a solid–solid interface.

Condition 1) occurs more frequently than 2). The displacement model formulation [15] yields prescribed stresses on the boundary [including 2) above] as a natural condition, and will be used here.[1] Of the two methods [14] of satisfying the remaining conditions, the method of forced constraints is the easier to implement. This is because constraints on the displacement components may be imposed simply by deleting (or ignoring) a row and column of the relevant element matrices. The alternative method required additional terms in the variational integral, and is therefore less attractive in the present formulation. Using the appropriate constraints on displacement components, leaving the other components unconstrained, and using the contour integrals in (4), it follows that conditions 2)–4) may now be satisfied.

On a solid–solid interface, all displacement components and the three stress components on the interface must all be continuous. In the displacement model, displacement continuity is already constrained. Furthermore, consideration of the integral along the interface from the contour integrals in (4) shows that stress on the interface is continuous as a natural condition for constrained displacement continuity. (This follows from the classical form of Hamilton's principle [14].)

It is noteworthy that the above conditions are less of a problem than in multiple dielectric EM guiding [12], where, for the usual formulation, the variational principle *must* be modified to accommodate the required interface conditions.

IV. Finite-Element Formulation

The popular displacement (or compatible) model is employed [15], and Silvester's notation is adopted where applicable [6].

A. Displacement Interpolation

Consider the rth triangular element, assumed to be entirely in medium m. Local area coordinates are defined for any point (x, y) in the triangle [5]

$$\zeta_i = (a_i + b_i x + c_i y)/2\Delta$$

where Δ is the triangle area; $b_i=y_j-y_k$; $c_i=x_k-x_j$; $a_i=x_jy_k-x_ky_j$; (x_i, y_i) are the coordinates of vertex i of the triangle; i ranges from 1 to 3; and i, j, and k are cyclic ($j=i+1$, $k=j+1$, mod 3). A set of n uniformly spaced points is defined over the triangle and ordered in the usual way [6], where $n=(N+1)(N+2)/2$ and $N\geq 1$ is an integer. Let the function u_x be represented by the discrete variables u_{xp} at the respective triangle points $p=1, \cdots, n$. The interpolated function for u_x then is

$$u_x(\zeta_1, \zeta_2, \zeta_3) = \sum_{p=1}^{n} \alpha_p(\zeta_1, \zeta_2, \zeta_3)u_{xp} \qquad (5)$$

where the α_p are Newton–Cotes interpolation polynomials of order N [6]. Similar expressions apply for u_y and u_z in terms of u_{yp} and u_{zp}, respectively. Partial x and y derivatives follow immediately, for example

$$\frac{\partial u_x}{\partial y} = \sum_{q=1}^{n} \sum_{i=1}^{3} \frac{c_i}{2\Delta} \frac{\partial \alpha_q}{\partial \zeta_i} u_{xq}$$

while a typical term encountered on taking the variation is

$$\frac{\partial}{\partial u_{xp}}\left(\frac{\partial u_x}{\partial y} \frac{\partial u_y}{\partial x}\right) = \sum_{q=1}^{n}\sum_{i=1}^{3}\sum_{j=1}^{3} \frac{b_j c_i}{4\Delta^2} \frac{\partial \alpha_p}{\partial \zeta_i} \frac{\partial \alpha_q}{\partial \zeta_j} u_{yq}.$$

The task of expanding $\delta L_r^{(m)}$ (the variation of the rth element) in terms of the discrete variables and taking the variation is indeed tedious, although straightforward in principle. When this is done, terms can be grouped to permit easy separation of geometric and material properties from terms independent of these. The main results of the formulation are given in the following paragraphs, omitting the detail of the development.

B. Fundamental Matrices

It is found that just five fundamental matrices are sufficient to represent $\delta L_r^{(m)}$ together with a set of simple coefficients. Three of these matrices each have three permutations equivalent to a redefinition of the triangle variables according to the vertex from which the element is viewed. The method of permutation is as described for Q matrices [6]. The dimensionless fundamental matrices are defined below for a typical element pq ($p, q=1, \cdots, n$):

$$M_{1pq} = \frac{1}{2\Delta}\int \alpha_p\alpha_q \, dS$$

$$M_{2pq}^{(i)} = \frac{1}{2\Delta}\int \left(\frac{\partial\alpha_p}{\partial\zeta_j} - \frac{\partial\alpha_p}{\partial\zeta_k}\right)\left(\frac{\partial\alpha_q}{\partial\zeta_j} - \frac{\partial\alpha_q}{\partial\zeta_k}\right) dS$$

$$M_{3pq} = \frac{1}{2\Delta}\int \sum_{i=1}^{3}\frac{\partial\alpha_p}{\partial\zeta_i}\left(\frac{\partial\alpha_q}{\partial\zeta_j} - \frac{\partial\alpha_q}{\partial\zeta_k}\right) dS$$

$$M_{4pq}^{(i)} = \frac{1}{2\Delta}\int \left(\alpha_q\frac{\partial\alpha_p}{\partial\zeta_i} + \alpha_p\frac{\partial\alpha_q}{\partial\zeta_i}\right) dS$$

$$M_{5pq}^{(i)} = \frac{1}{2\Delta}\int \left(\alpha_q\frac{\partial\alpha_p}{\partial\zeta_i} - \alpha_p\frac{\partial\alpha_q}{\partial\zeta_i}\right) dS \qquad (6)$$

where the superscripts refer to permutation numbers; the i, j, and k are cyclic (mod 3); and integration is taken over the surface of the finite element.

The first three of these have appeared in other contexts in the literature: $[M_1]$ and $[M_2]^{(1)}$ are directly related to Silvester's T and Q_1 matrices;[2] and $[M_3]$ is the coupling matrix presented by Daly [12]. The remaining two are necessary to

[1] Other formulations [15], [17], [18] would require either additional terms in the variational integral or constrained stress components on traction-free boundaries, but could equally well be used.

[2] $[M_1]$ is obtained by halving the T matrix, while $[M_2]^{(1)}$ is just the Q_1 matrix, both as presented in [6, appendix], which differs in minor respects from the definitions of T and Q_1.

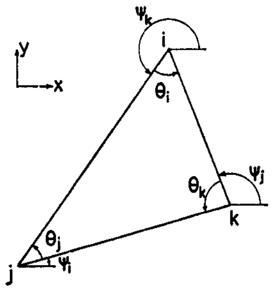

Fig. 1. Conventions used for included and inclined angles θ and ψ for a triangular element.

represent the β-dependent terms in $\delta L_r^{(m)}$ and have been evaluated for up to fourth-order elements [9]. These matrices define the coupling between the u_x and u_z field and between the u_y and u_z field for traveling-wave solutions.

C. Geometric Matrices

A set of geometric matrices, which arise from the grouping of terms in $\delta L_r^{(m)}$, can now be defined. They are expressed in terms of (6) and coefficients related to the geometry of the element under consideration. The geometric matrices are also dimensionless, and are defined as follows:

$$\begin{aligned}
[G_1] &= (2\Delta/h^2)[M_1] \\
[G_2] &= \sum_i \cot\theta_i [M_2]^{(i)} \\
[G_3] &= \sum_i \frac{\cos(\psi_j+\psi_k)}{\sin\theta_i}[M_2]^{(i)} \\
[G_4] &= \sum_i \frac{\sin(\psi_j+\psi_k)}{\sin\theta_i}[M_2]^{(i)} \\
[G_5] &= [M_3] \\
[G_6] &= \sum_i (b_i/h)[M_4]^{(i)} \\
[G_7] &= \sum_i (b_i/h)[M_5]^{(i)} \\
[G_8] &= \sum_i (c_i/h)[M_4]^{(i)} \\
[G_9] &= \sum_i (c_i/h)[M_5]^{(i)}
\end{aligned} \tag{7}$$

where the summations are taken for $i=1, \cdots, 3$; i, j and k are cyclic (mod 3); and h is any convenient dimension of the profile being analyzed. ψ_i is the inclination of the side opposite vertex i to the x axis and θ_i is the interior angle at vertex i (Fig. 1).

At this point it may be noted that the formulation must be orientation dependent with respect to rotation about the z axis. This is because the components u_x and u_y are themselves orientation dependent, in contrast to the EM case which is formulated only in terms of z-directed components [6], [12].

D. Element Matrix Assembly

The expansion of (3) reveals that there are terms both in β and β^2. This suggests that normalized phase "constant" $\bar{\beta}=\beta h$ could best be used as the independent variable in the determination of the modal spectrum leaving the squared normalized phase velocity $\nu=v^2\rho^{(1)}/\mu^{(1)}$ to be extracted as the eigenvalue. Using this scheme, the element variation is approximated by

$$\delta L_r^{(m)} \simeq \mu^{(m)}([A_r] - \nu[B_r])[U_r] \tag{8}$$

where

$$[U_r] = [\{u_{xp}\} \mid \{u_{yp}\} \mid \{u_{zp}\}]^t, \qquad p = 1, \cdots, n.$$

The element matrices $[A_r]$ and $[B_r]$ divide naturally into the partitions by columns, as indicated by the partitioning of $[U_r]$, and by rows according to the equations arranged in the order

$$\left\{\frac{\partial L_r^{(m)}}{\partial u_{xp}}\right\}, \left\{\frac{\partial L_r^{(m)}}{\partial u_{yp}}\right\}, \left\{\frac{\partial L_r^{(m)}}{\partial u_{zp}}\right\}, \qquad p = 1, \cdots, n.$$

The result is that $[A_r]$ and $[B_r]$ are both symmetric, and the latter is also positive definite.

Finally, the partitions $[P_A]_{ij}$ and $[P_B]_{ij}$ of $[A_r]$ and $[B_r]$, respectively, where $i, j=1, \cdots, 3$, may now be written in terms of (7), where symmetry requires only six of each set to be defined:

$$\begin{aligned}
[P_A]_{11} &= \bar{\beta}^2[G_1] + C_1[G_2] + C_2[G_3] \\
[P_A]_{12} &= C_2[G_4] + C_3[G_5] \\
[P_A]_{13} &= \bar{\beta}(C_3[G_6] + C_2[G_7]) \\
[P_A]_{22} &= \bar{\beta}^2[G_1] + C_1[G_2] - C_2[G_3] \\
[P_A]_{23} &= \bar{\beta}(C_3[G_8] + C_2[G_9]) \\
[P_A]_{33} &= \bar{\beta}^2 R^{(m)}[G_1] + [G_2] \\
[P_B]_{11} &= [P_B]_{22} = [P_B]_{33} = \bar{\beta}^2(v_s^{(1)}/v_s^{(m)})^2[G_1] \\
[P_B]_{12} &= [P_B]_{13} = [P_B]_{23} = 0
\end{aligned} \tag{9}$$

where $v_s^{(m)}=(\mu^{(m)}/\rho^{(m)})^{1/2}$ is the bulk shear velicity in the mth medium, $C_1=(R^{(m)}+1)/2$, $C_2=(R^{(m)}-1)/2$, $C_3=(R^{(m)}-3)/2$, and $R^{(m)}=(\lambda^{(m)}+2\mu^{(m)})/\mu^{(m)}$.

Summation of the element variations (8) is performed in the usual manner [13], and the matrix-eigenvalue/eigenvector problem is solved by standard methods.

E. Computer Programming

The most significant feature of the above formulation is that it requires only minimal computer coding for its implementation. To illustrate this, assume that node coordinates and interconnections are provided and that a compatible set of variables has been generated for each triangle (or read from data). In a prototype program, which accepts any problem allowed by the formulation, the complete assembly of the matrices was effected in 150 Fortran statements (including four supporting routines for the permutation and manipulation of matrices). The time taken for this assembly is always insignificant compared to the solution time for the matrix eigenvalue–eigenvector problem.

V. Examples

Testing of the formulation is a problem because there are few acoustic propagation problems amenable to exact solution. Good agreement with some experimental results was reported in [9] for a homogeneous ridge guide. Three examples will be given here—a more exact test for a homogeneous problem, a layered plate with straight crested solutions, and a nonhomogeneous surface acoustic waveguide on a finite substrate.

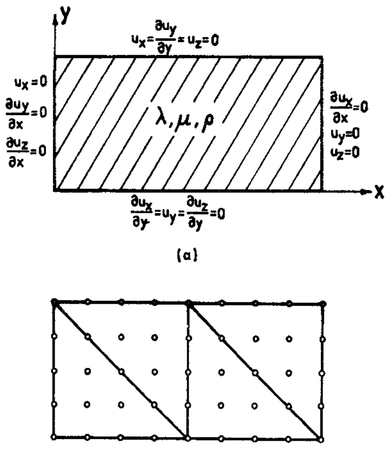

Fig. 2. Homogeneous rectangular structure. (a) Two-dimensional cross section showing boundary conditions on displacement components. (b) Fourth-order element discretization.

A. Homogeneous Rectangular Region

Waldron [20] has pointed out that the "cardinal" surfaces encountered at physical boundaries of elasticity problems preclude separable solutions. However, it is a simple matter to construct problems, which may not be physical, having as boundary conditions a combination of the symmetry conditions outlined in Section III. Such problems can yield separable solutions. Consider the homogeneous isotropic rectangular region $0<x<1$, $0<y<0.5$, bounded by symmetry conditions on $x=0$ and $y=0$ and antisymmetry conditions on $x=1$ and $y=0.5$ [Fig. 2(a)]. Putting $h=0.5$ and

$$\begin{aligned} u_x &= D \sin k_x x \cos k_y y \\ u_y &= E \cos k_x x \sin k_y y \\ ju_z &= jF \cos k_x x \cos k_y y \end{aligned} \tag{10}$$

it follows that the prescribed boundary conditions will be satisfied if

$$k_x = p\frac{\pi}{2} \qquad k_y = q\pi$$

where p and q are odd integers. Substitution of (10) into the displacement wave equations results in a 3×3 matrix eigenvalue problem, which has been solved for the displacement amplitudes in (10) and phase velocity, for $\bar{\beta}=1.5$ and Poisson's ratio $=\frac{1}{3}$. The problem has also been solved using the fourth-order finite-element discretization shown in Fig. 2(b). The first few eigenvalues obtained from the exact solution and corresponding percentage deviations in the finite element solutions are summarized in Table I. Agreement is close and is better than 1 percent for the first 13 modes for the discretization used. Table II shows the displacement field amplitudes for mode 1, again showing the close agreement.

B. Layered Plate

Consider an infinite plate of thickness $2s$, layered on both sides by a second material of thickness h (Fig. 3). The two limiting cases $s/h=0$ (infinite plate, thickness $2h$) and $s/h\to\infty$ (layered half space) have known solutions. The fundamental Rayleigh-type mode for this structure has been analyzed using the above formulation by constraining the u_x field to zero and analyzing a narrow strip to enforce straight crested solutions. Dispersion curves obtained for various s/h using this procedure are plotted in Fig. 4 (dashed curves). Good agreement is obtained with the limiting cases: Tolstoy and Usdin's [21] layered half-space calculation (whose material constants were used), and with plate modes, calculated from equations in [22]. The gradient of the curve at $\bar{\beta}=0$ (dotted line, Fig. 4) was calculated from Tiersten [2, eq. 4.15], allowing for the different normalization used there. Excellent agreement has also been obtained with Tiersten's gold on fused quartz layered half-space calculation. In every

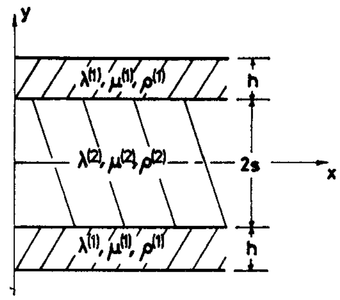

Fig. 3. Dimensions of the layered plate, infinite in the x direction and symmetric about the x axis.

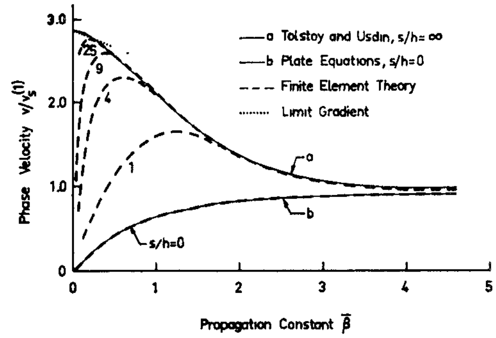

Fig. 4. Phase-velocity dispersion curves for the fundamental antisymmetric mode of the layered plate, with s/h as parameter, $\mu^{(2)}/\mu^{(1)}=13.77$, $\rho^{(2)}/\rho^{(1)}=1.39$, and Poisson's ratio $=0.25$ in each medium. Full curves are a—Tolstoy–Usdin theory [21], and b—calculated from plate equations. Dashed curves are for finite-element theory and dotted line is limit gradient at $\bar{\beta}=0$, calculated from Tiersten [2].

TABLE I

EIGENVALUES FOR RECTANGULAR HOMOGENEOUS REGION

Mode	pq	Exact Eigenvalue $\left[\frac{v}{v_s}\right]^2$	Finite Element Soln. % deviation
1	11	9.483114	-0.00009
2	11	2.37078	0.00041
3	11	2.37078	0.0011
4	31	18.2561	0.0061
5	31	4.5640	0.03
6	31	4.5640	0.06

Note: $\bar{\beta}=1.5$; Poisson's ratio $=\frac{1}{3}$.

TABLE II

EIGENVECTOR FOR MODE 1, TABLE I

Displacement Amplitude	Finite Element Solution
D 0.5000	0.4992
E 1.0000	1.0000
F 0.9549	0.9546

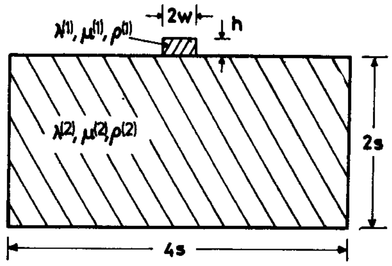

Fig. 5. Geometry of rectangular gold overlay on a rectangular fused quartz substrate.

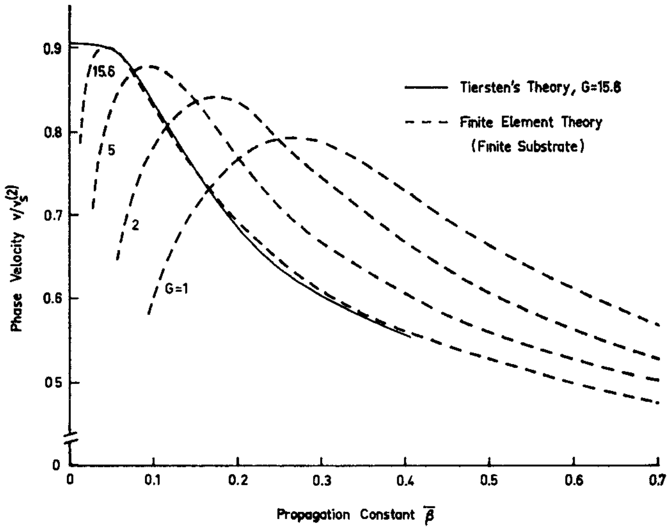

Fig. 6. Dispersion curves for the gold overlay on a fused quartz substrate having the geometry of Fig. 5 for $s/w=10$ and with $G=w/h$ as parameter. Tiersten's result for $G=15.6$ on an infinite substrate is shown (full curve) for comparison with finite-element theory (dashed curves).

case, the displacement functions were very nearly straight crested, as required.

C. Overlay Guide

The ability of an overlay of gold on fused silica to guide surface acoustic waves was first reported by White [19] and analyzed by Tiersten [2]. Consider such a guide on a practical (finite) substrate having the dimensions indicated in Fig. 5. Dispersion characteristics have been obtained for the first mode symmetric about the bisector of the overlay for $s/w=10$ and various $G=w/h$ (Fig. 6).

The curves agree closely with other analyses of the *unbounded* problem [2], [23], except at low frequencies where the transition from the guide mode to the first antisymmetric mode of the substrate is clearly evident. Below this transition frequency, displacement fields show that the mode is entirely characteristic of the substrate, and therefore cannot be said to be guided by the overlay. Analysis shows that the vertical sides of the substrate have little effect on the curves of Fig. 6. It follows that low-frequency performance for $s/w \neq 10$ can be estimated using the linear relationship between infinite plate characteristics and plate thickness.

VI. Conclusions

The finite-element formulation presented enables a large class of acoustic propagation problems to be solved. In principle, modes of propagation may be determined for any structure whose cross section consists of regions of lossless homogeneous isotropic media in rigid contact and uniform in the direction of propagation Only minimal computer coding is required to set up finite elements of any desired order. The three examples cited demonstrate the accuracy and versatility of the method and represent a comprehensive test of the formulation, showing close agreement with other analyses.

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